EXPONENTIAL SUMS OVER PRIMES WITH MULTIPLICATIVE COEFFICIENTS

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Abstract. We consider exponential sums of the form

$$\sum_{X<p\leq 2X} f(p)(\log p)e(p\alpha)\ ,$$

where the sum runs over the prime numbers $p\in(X,2X]$ and f is a multiplicative function satisfying certain growth conditions. As a consequence of our result, we consider the normalized Fourier coefficients $(a_g(n))$ of any eulerian $GL(n)$-cuspform g that satisfies the Ramanujan conjecture as well as an estimate of the form $\max_{\alpha\in\mathbb{R}}|\sum_{n\leq X}a_g(n)e(n\alpha)|\leq X^\eta$ for some $\eta<1$. For such a form, we get that

$$\sum_{X<p\leq 2X} a_g(p)(\log p)e(p\alpha)\ll\frac{\sqrt{q}}{\varphi(q)}X\ ,$$

where α is a real number such that $\left|\alpha-\frac{a}{q}\right|\ll X^{-1+\frac{1-\eta}{120}}$ for some $q\leq X^{(1-\eta)/15}$. Under stronger restrictions and the same conditions on α and a/q, we also prove that

$$\sum_{X<\ell\leq 2X} a_g(\ell)\mu(\ell)e(p\alpha)\ll X/\sqrt{q}\ .$$

1. Introduction

We are concerned here in getting estimates for the trigonometric polynomial $\sum_{p\leq N}\tau(p)e(p\alpha)$ where p is a prime number and τ the Ramanujan function. We consider more precisely phases α that are close to a rational a/q with a small denominator q. It can be seen in the general light of Sarnak's conjecture [15] as showing that $\tau(p)$ do not correlate with the additive characters we consider, continuing the work [3] of É. Fouvry and S. Ganguly.

Our method fits in a more general framework and continues our previous query [12] where we obtained 'optimal' bounds for $\sum_{p\leq N}e(p\alpha)\log p$. We add a multiplicative function f as a coefficient to consider $\sum_{p\leq N}f(p)e(p\alpha)\log p$ and want to show that this sum is small provided reasonable hypotheses on the values of f *over the integers* are met. We present these assumptions just below, but we want to stress out that the main difficulty with respect to our previous work is that we do not assume $|f(p)|\leq 1$.

Let us present the three assumptions we make on our multiplicative function f.

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(H_1) There exists an integer $k \geq 1$ such that

$$|f(n)| \leq \tau_k(n) \quad \text{for all } \; n \geq 1\,.$$

Here and in what follows τ_k denotes the kth divisor function defined by $\tau_k(n) = \sum_{d_1 \cdots d_k = n} 1$. We fix such an integer k once and for all.

(H_2) On denoting by f^{-1} the Dirichlet inverse of f, we assume that for each $\epsilon > 0$,

$$|f^{-1}(n)| \ll_\epsilon n^\epsilon.$$

(H_3) There exists an η ($\frac{1}{2} \leq \eta < 1$) such that

$$\sum_{n \leq X} f(n) e(n\alpha) \ll X^\eta$$

uniformly in $\alpha \in \mathbb{R}$.

These hypotheses have for instance been investigated in the context of automorphic forms and we describe this situation some more below. The class $\mathcal{C}(k,\eta)$ of functions satisfying (H_1), (H_2) and (H_3) could be extended somewhat, but we keep it for simplicity. Notice that the convolution product of two multiplicative functions, one from $\mathcal{C}(k_1,\eta_1)$ and the other from $\mathcal{C}(k_2,\eta_2)$, belongs to $\mathcal{C}\Big(k_1+k_2, \frac{1-\eta_1\eta_2-(\eta_1-\eta_2)^2}{2-\eta_1-\eta_2}\Big)$ (this is a simple consequence of the Dirichlet hyperbola formula). Broadly speaking, the question we address is to infer properties on the primes from properties on the integers as in sieve theory. Here is our first result, where φ denotes the Euler's function.

Corollary 1.1. *Assume (H_1), (H_2) and (H_3). Let $X \geq 1$ be a real number. Let q be a positive integer such that $q \leq X^{(1-\eta)/15}$. Then for any real number α with $|\alpha - \frac{a}{q}| \ll X^{-1+\frac{1-\eta}{120}}$ we have*

$$\sum_{X < \ell \leq 2X} f(p)(\log p) e(p\alpha) \ll_\eta \frac{\sqrt{q}X}{\varphi(q)}\,.$$

The constant implied in the $\ll_\eta$-symbol depends only on the constants implied in hypotheses (H_1), (H_2) and (H_3).

This is a direct corollary of Theorem 1.4 below. Our proof goes by building in Section 3 (more precisely (19)) *a family* of bilinear representations of the characteristic function of the primes, a task for which we rely heavily on the prior work of Y. Motohashi in [10] (see also [9]). The averaging effect on this family will save the last $\log X$.

Examples of the situation we consider is given by non-trivial Dirichlet characters; more interesting situations are given by cuspforms. For instance, we can consider a normalized Hecke eigen cuspform g of weight m on $SL_2(\mathbb{Z})$ and its sequence $\{a_g(n)\}$ of normalized Fourier coefficients (this means for instance that rather than considering the Ramanujan τ function, we prefer to investigate $\tau(n)n^{-11/2}$). Then it is known that a_g is a multiplicative function that belongs to $\mathcal{C}(2, 1/2)$. Indeed, Deligne's bound says that a_g satisfies (H_1) with $k = 2$. It can also be seen, looking at the Euler product of L-function associated to g, that the function a_g satisfies (H_2). And it is proved by M. Jutila [4] that (H_3) is satisfied by a_g with $\eta = \frac{1}{2}$, refining the work of J.R. Wilton in [18, Lemma 3]. Assumption (H_1) in this context would be a consequence of the Ramanujan Conjecture, which is known to hold in several cases. A similar setting can be developped for $GL(n)$-cuspforms; Assumption (H_1) would be a consequence of the proper Ramanujan Conjecture, Assumption (H_2) would be given by the rationality of the Euler-factor of the corresponding Dirichlet series. We refer to Chapter 9 by J.W. Cogdell from the book [1] for the theory of $GL(n)$-L-functions and in particular the eulerianity of Fourier expansions. These two 'size' hypotheses are somewhat stronger than the corresponding ones used for defining the Selberg class of [16]. Assumption (H_3) has been investigated by S.D. Miller in [8] for $GL(3)$-forms and is believed to hold in general. The reader will find a discussion of this in the paper [7] by Guangshi Lü.

Hence by Corollary 1.1 we have the following.

Corollary 1.2. *Let g be an eulerian $GL(n)$-cuspform that satisfies the Ramanujan conjecture and (H_3) for some $\eta < 1$. Let q be a positive integer such that $q \leq X^{(1-\eta)/15}$. Then for any real number α with $|\alpha - \frac{a}{q}| \ll X^{-1+\frac{1-\eta}{120}}$ we have*

$$\sum_{X<\ell\leq 2X} a_g(p)(\log p)e(p\alpha) \ll \frac{\sqrt{q}X}{\varphi(q)} .$$

Let us now be more precise. We introduce $L(f,s)$ (for $\Re(s) > 1$) the L-function associated to f, i.e.

$$L(f,s) = \sum_{n\geq 1} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right) .$$

Let $\frac{L'(f,s)}{L(f,s)} = -\sum_n \Lambda_f(n)n^{-s}$ be the logarithmic derivative of $L(f,s)$, see [6]. Then clearly Λ_f has support only on prime powers. In fact we have:

Lemma 1.3. *Let p be a prime number. For any integer $m \geq 1$ we have*

$$\Lambda_f(p^n) = \sum_{\substack{k+\ell=n \\ \ell\geq 1}} f^{-1}(p^k)f(p^\ell)\log(p^\ell) .$$

From (H_1) and (H_2) together with the above lemma and the well-known $\tau_k(n) \ll_\epsilon n^\epsilon$, we get

$$|\Lambda_f(p^n)| \ll_\epsilon p^{n\epsilon} . \tag{1}$$

In view of this bound we have

$$\sum_{\substack{X<p^m\leq 2X \\ m\geq 2}} |\Lambda_f(p^m)| \ll_\epsilon X^{\frac{1}{2}+\epsilon} \tag{2}$$

for each $\epsilon > 0$. Here is the main theorem of this paper when expressed in terms of Λ_f.

Theorem 1.4. *Let $X \geq 1$ be a real number and let f be a multiplicative function satisfying (H_1), (H_2) and (H_3). Let q be any positive integer such that $q \leq X^{(1-\eta)/15}$. Then for any real number α with $|\alpha - \frac{a}{q}| \ll X^{-1+\frac{1-\eta}{120}}$ we have*

$$\sum_{X<\ell\leq 2X} \Lambda_f(\ell) e(\ell\alpha) \ll_\eta \frac{\sqrt{q}X}{\varphi(q)} .$$

Since $\Lambda_f(p) = f(p)\log p$ for all primes p, Corollary 1.1 can be easily obtained on using (2). We did not try to optimize the exponents that appear but only aimed at producing a clean proof.

The method of proving Theorem 1.4 is flexible enough to obtain similar results for the sum of f^{-1} over square-free integers. We however need to control $\sum_{n\leq M}|f(n)|^2$. Concerning holomorphic modular form, this (and more) follows from the work [13] of Rankin. See also [2] by O.M. Fomenko and [5] by H. Lao for symetric powers L-functions.

Theorem 1.5. *Let $X \geq 1$ be a real number and let f be a multiplicative function satisfying (H_1), (H_2) and (H_3). Let q be any positive integer such that $q \leq X^{(1-\eta)/15}$. Then for any real number α with $|\alpha - \frac{a}{q}| \ll X^{-1+\frac{1-\eta}{120}}$ we have*

$$\sum_{X<\ell\leq 2X} \mu(\ell) f(\ell) e(\ell\alpha) \ll_\eta \frac{X}{\sqrt{q}}\sqrt{W_f(X)} \tag{3}$$

where

$$W_f(X) = \max_{M\leq X} \sum_{M<m\leq 2M} |f(m)|^2/M . \tag{4}$$

Let us note here that, if considering convolutions in Theorem 1.4 is rather pointless as the corresponding value at the primes is simply the sum of the one of each factor. The situation changes drastically here. As already mentioned, Hypothesis (H_3) in this contexct has been investigated by Guangshi Lü in [7]. Since the Möbius function satisfies our hypotheses under the Generalized Riemann Hypothesis, we see that the above theorem is conjecturally optimal at least when $q = 1$ and $\alpha = 0$. The coefficient $W_f(X)$ is introduced to accomodate possible powers of $\log X$. Notice that

$$\sum_{M<m\leq 2M} |f(m)| \leq 2\sqrt{W_f(X)}M \tag{5}$$

whenever $M \leq X$.

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Organisation of the proof. The proof starts with Equation (27) which proposes a decomposition of the relevant trigonometric sum in three parts. The first part, $L_r^{(1)}(\alpha)$, is studied in Section 4, while the second one is studied in Section 5. These are technical but rather straightforward as no particular precision is required. The third and last part, which we call the *bilinear sum* is handled in Section 6. The sum S^b in (6) is the central quantity and contains the averages over r (i.e. the averages over the family bilinear decompositions). We first need to reduce the evaluation to an L^2-problem, then we localize the variables and finally we separate them properly (they are linked with a mild $X < mn \leq 2X$). At the same time, we take care of the offset between α and a/q. This introduces the error terms $E_1(\delta, r)$, $E_2(\delta, r)$ and $\delta E_3(r)$. It is then a matter of bookkeeping to reduce the expression obtained to the hybrid large sieve inequality we have recalled earlier in Lemma 2.7. The proof of Theorem 1.5 is mutatis mutandis to the proof of Theorem 1.4. We give a sketch of it in Section 7.

2. Preliminaries

We state the following lemma of Motohashi, see [10, page 25].

Lemma 2.1. *Let d be a square-free integer. Then for any positive integer n, we have*

$$f(dn) \;=\; \mu(d) \sum_{u|n,\; u|d^\infty} f\left(\frac{n}{u}\right) f^{-1}(du)\;.$$

Here $u \mid d^\infty$ means that u divides some power of d.

Lemma 2.2. *Let ℓ and d be positive integers. Then for each $\epsilon > 0$ we have*

$$\sum_{\substack{n\geq 1\\ n|[\ell,d]^\infty}} \frac{f^{-1}([\ell,d]n)}{n^{\frac{1}{2}}} \ll_\epsilon (d\ell)^\epsilon\;.$$

Let $\theta > 0$ (we shall finally choose $\theta = 1/k^2$) and $z > 1$ be fixed. For any positive integers k' and d, define $\Lambda_d^{(k')}$ by

$$\Lambda_d^{(k')} \;=\; \frac{1}{k'!}(\theta \log z)^{-k'} \sum_{j=0}^{k'} (-1)^{k'-j} \binom{k'}{j} \lambda_d^{(j,k')}\;,$$

where

$$\lambda_d^{(j,k')} \;=\; \begin{cases} \mu(d)\left(\log \frac{z^{1+j\theta}}{d}\right)^{k'} & \text{if } d < z^{1+j\theta};\\ 0 & \text{otherwise.}\end{cases}$$

We have the following lemma.

Lemma 2.3 (Motohashi, Theorem 4, [10])**.** *The weights $\Lambda_d^{(k')}$ satisfy the following:*

1. *$\Lambda_d^{(k')} = \mu(d)$ if $d < z$.*
2. *Let $c > 0$ be some parameter. We have $\sum_{n\geq 1} \tau_{k'}(n) \left(\sum_{d|n} \Lambda_d^{(k')}\right)^2 n^{-\omega} \ll 1$, provided $\omega \geq 1 + \frac{c}{\log z}$.*

The weights $\Lambda_d^{(k')}$ further satisfy the following bound.

Lemma 2.4. *Let $\theta > 0$. Then*

$$|\Lambda_d^{(k')}| \le \frac{(2k')^{k'}}{k'!} \ .$$

Proof. Since the quantity on the right hand side is greater than 1, we can assume that $d \ge z$. Then we have

$$|\Lambda_d^{(k')}| \le \frac{1}{k'!(\theta \log z)^{k'}} \sum_{j=0}^{k'} \binom{k'}{j} (\log z^{j\theta})^{k'} \ .$$

Cancelling $(\log z^{\theta})^{k'}$, using $j \le k'$ and $\sum_{j=0}^{k'} \binom{k'}{j} = 2^{k'}$ gives the desired upper bound. □

The function f being given to verify assumptions (H_1), (H_2) and (H_3). They imply the two parameters k and η that we keep fixed throughout the proof. We use the above weights with $k' = k^2$ and some $\theta > 0$ that we keep as a parameter until the end of the proof where we choose $\theta = \frac{1}{k^2}$. For any $r \ge 1$, let $M_r(s)$ be defined by

$$M_r(s) \ = \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell | r} \frac{\ell \mu\left(\frac{r}{\ell}\right)}{[\ell, d]^s} \mu([\ell, d]) \sum_{n \ge 1} \frac{f^{-1}_{[\ell,d]}(n)}{n^s}, \tag{6}$$

where

$$f^{-1}_{[\ell,d]}(n) = \begin{cases} f^{-1}([\ell, d]n) \text{ when } n \mid [\ell, d]^{\infty}; \\ 0 \quad \text{otherwise} \end{cases}$$

Lemma 2.5. *Let $\Re(s) \ge \frac{1}{2}$. Then for each $\epsilon > 0$ we have*

$$M_r(s) \ll_{\epsilon} r^{1+\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} \ .$$

Proof. We have for $\Re(s) \ge \frac{1}{2}$,

$$\begin{aligned} |M_r(s)| &\le \sum_{d \le z^{1+k^2\theta}} |\Lambda_d^{(k^2)}| \sum_{\ell | r} \frac{\ell}{[\ell, d]^{\frac{1}{2}}} \sum_{n \ge 1} \frac{|f^{-1}_{[\ell,d]}(n)|}{n^{\frac{1}{2}}} \\ &\ll_k \sum_{d \le z^{1+k^2\theta}} \frac{1}{d^{\frac{1}{2}}} \sum_{\ell | r} \ell \sum_{\substack{n \ge 1 \\ n | [\ell,d]^{\infty}}} \frac{f^{-1}([\ell, d]n)}{n^{\frac{1}{2}}} \ . \end{aligned}$$

We have by Lemma 2.2

$$\sum_{\substack{n \ge 1 \\ n | [\ell,d]^{\infty}}} \frac{f^{-1}([\ell, d]n)}{n^{\frac{1}{2}}} \ll_{\epsilon} (\ell d)^{\epsilon} \ .$$

From this we get

$$|M_r(s)| \ll_{\epsilon} \sum_{d \le z^{1+k^2\theta}} \frac{1}{d^{\frac{1}{2}-\epsilon}} \sum_{\ell | r} \ell^{1+\epsilon} \ .$$

Hence

$$M_r(s) \ll_{\epsilon,k} r^{1+2\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} .$$

□

From (H_3) we obtain the following estimate by partial summation.

$$\sum_{n\le X} f(n)(\log n)e(n\alpha) \ll X^\eta \log X . \tag{7}$$

Now we state one of the crucial lemmas.

Lemma 2.6 (Lemma 41, [12])**.** *Let $\delta \in (0,1/2)$, β and $X \ge 1$ be three real parameters. There exists a C^1-function $\mathcal{H}$ such that for any sequence (a_ℓ) of complex numbers, we have*

$$\sum_{X<\ell\le 2X} a_\ell\, e(\beta\ell) = \int_{-\Delta}^{\Delta} \sum_{\ell\ge1} \frac{a_\ell}{\ell^{iu}} \mathcal{H}(u) X^{iu} du + \mathcal{O}^*\Bigg(\sum_{\substack{X<\ell\le(1+\delta)X,\\ or\ (2-\delta)X<\ell\le 2X}} |a_\ell| + 2\delta \sum_{\ell\ge1} |a_\ell| \Bigg) ,$$

where $\Delta = 100\frac{\delta^{-1}+(\beta X)^2}{\delta}$. We have furthermore $|\mathcal{H}(u)| \le \frac{25}{73}(1+|\beta|X)/(1+|u|)$ and $\int_{-\infty}^{\infty} |\mathcal{H}(u)|^2 du = (\log 2)^2(2-2\delta)/(4\pi)^2$.

For any positive integers r and n, the Ramanujan sum $c_r(n)$ is defined by

$$c_r(n) \;=\; \sum_{a (\mathrm{mod})^* r} e\left(\frac{an}{r}\right) , \tag{8}$$

where the sum runs over all the coprime residue classes modulo r. It is known that $c_r(n)$ can also be expressed as

$$c_r(n) \;=\; \sum_{\ell|n, \ell|r} \ell\mu\left(\frac{r}{\ell}\right).$$

By definition of Ramanujan sum it is clear that $|c_r(n)| \le \varphi(r)$. Now we state the following version of Large sieve inequality:

Lemma 2.7 (Theorem 10, [12])**.** *Let q be some fixed modulus and N_0 be some real number. Let $(u_n)_n$ be a sequence of complex numbers that is such that $\sum_n(|u_n| + n|u_n|^2) < \infty$. Then we have, for any $T \ge 0$,*

$$\sum_{\substack{r\le R/q,\\ (q,r)=1}} \frac{1}{\varphi(r)} \sum_{a \bmod q} \int_{-T}^{T} \Big|\sum_n u_n c_r(n+N_0) n^{it} e(na/q)\Big|^2 dt$$

$$\le 7 \sum_n |u_n|^2 (n + R^2 \max(T,10)).$$

Let us recall the classical definition

$$G_q(D) = \sum_{\substack{d\le D,\\ (d,q)=1}} \frac{\mu^2(d)}{\varphi(d)}, \quad G(D) = G_1(D). \tag{9}$$

We quote from [17]:

$$G(D) \le \frac{q}{\varphi(q)} G_q(D) \le G(qD). \tag{10}$$

We quote from [11, Lemma 3.5] (see also [14])

$$G(D) \le \log D + 1.4709, \quad (D \ge 1) \tag{11}$$

and, concerning a lower bound,

$$\log D + 1.06 \le G(D), \quad (D \ge 6). \tag{12}$$

For any given integer $r \ge 1$, we define

$$\nu_r(n) \;=\; f(n) c_r(n) \left(\sum_{d|n} \Lambda_d^{(k^2)} \right). \tag{13}$$

We have the following estimate.

Lemma 2.8. *Let $B \ge 1$ be a real number. Then*

$$\sum_{r \le \frac{R}{q}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n \le B} \frac{|\nu_r(n)|^2}{n} \;\ll\; B^{\frac{c}{\log z}} \left(\frac{R}{q} \right)^2$$

for each $c > 0$.

Proof. By definition of $\nu_r(n)$, we have

$$\sum_{r \le \frac{R}{q}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n \le B} \frac{|\nu_r(n)|^2}{n} \;=\; \sum_{r \le \frac{R}{q}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n \le B} \frac{f^2(n) |c_r^2(n)| \left(\sum_{d|n} \Lambda_d^{(k^2)} \right)^2}{n} .$$

Let $c > 0$ be a real number. Then

$$\sum_{r \le \frac{R}{q}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n \le B} \frac{|\nu_r(n)|^2}{n} \;\le\; \sum_{r \le \frac{R}{q}} \mu^2(r) \varphi(r) B^{\frac{c}{\log z}} \sum_{n \le B} \frac{f^2(n) \left(\sum_{d|n} \Lambda_d^{(k^2)} \right)^2}{n^{1+\frac{c}{\log z}}} .$$

By Lemma 2.3 the last sum is bounded since $f^2(n) \le \tau_k^2(n) \le \tau_{k^2}(n)$, and hence

$$\sum_{r \le \frac{R}{q}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n \le B} \frac{|\nu_r(n)|^2}{n} \;\le\; B^{\frac{c}{\log z}} \left(\frac{R}{q} \right)^2 .$$

□

The following lemma will be used in Section 6.

Lemma 2.9. *Let M be a sufficiently large real number and q be a positive integer. Then*

$$\sum_{b (mod) q} \left| \sum_{\substack{m \equiv b(q) \\ m \sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 \ll \frac{M^2}{\varphi(q)} . \tag{14}$$

Proof. We have

$$\sum_{b \mod q} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 = \sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 + \sum_{\substack{b \mod q\\ (b,q)>1}} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 . \tag{15}$$

We have

$$\sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 \ll \sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{p\equiv b(q)\\ p\sim M}} \frac{\Lambda_f(p)}{p^{iu}} \right|^2 + \sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{p^t\equiv b(q)\\ p^t\sim M\\ t\geq 2}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 .$$

For the first sum we use $|\Lambda_f(p)| \leq k \log p$ after applying Cauchy-Schwarz inequality. The sum inside the modulus in the second sum is trivially $\ll M^{1+\epsilon}$ in view of (2) . Hence we have

$$\sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 \ll \frac{M}{\varphi(q)\log M} \sum_{b(\mathrm{mod})^* q} \sum_{\substack{p\equiv b(q)\\ p\sim M}} \log^2 p + M^{1+\epsilon}\varphi(q) .$$

Hence by prime number theorem we have

$$\sum_{b(\mathrm{mod})^* q} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 \ll \frac{M^2}{\varphi(q)}$$

Similarly we can show that the second sum on the right hand side of (15) is $\ll \frac{M^2}{\varphi(q)}$. □

3. Bilinear decomposition of Λ_f and μf

For any square-free integer $r \geq 1$, let

$$V_r(s) = \sum_{n\geq 2} \frac{\nu_r(n)}{n^s} ,$$

where $\nu_r(n)$ be as in (13). By Lemma 2.3 we can see that $\nu_r(n) = 0$ if $n \leq z$.

We have

$$\begin{aligned} 1 + V_r(s) &= 1 + \sum_{n\geq 2} \frac{f(n)c_r(n)}{n^s} \left(\sum_{d|n} \Lambda_d^{(k^2)} \right) \\ &= \sum_{d\leq z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell|r} \frac{\ell\mu\left(\frac{r}{\ell}\right)}{[\ell,d]^s} \sum_{n\geq 1} \frac{f([\ell,d]n)}{n^s} . \end{aligned}$$

By Lemma 2.1 the above identity becomes

$$1+V_r(s)=\sum_{d\le z^{1+k^2\theta}}\Lambda_d^{(k^2)}\sum_{\ell|r}\frac{\ell\mu\left(\frac{r}{\ell}\right)}{[\ell,d]^s}\mu([\ell,d])\sum_{n\ge1}\frac{f*f^{-1}_{[\ell,d]}(n)}{n^s}$$

$$=L(f,s)\sum_{m\ge1}\frac{h_r(m)}{m^s}\ . \tag{16}$$

Let $M_r(s)=\sum_{m\ge1}\frac{h_r(m)}{m^s}$. Write

$$1=-V_r(s)+(1+V_r(s))\ . \tag{17}$$

Multiplying both sides by $-\frac{L'(f,s)}{L(f,s)}$ we get

$$-\frac{L'(f,s)}{L(f,s)}=\frac{L'(f,s)}{L(f,s)}V_r(s)-L'(f,s)M_r(s)\ . \tag{18}$$

This gives the following decomposition for $\Lambda_f(n)$:

$$\Lambda_f(n)=-(\Lambda_f*\nu_r)(n)+(f\log*h_r)(n)\ . \tag{19}$$

Define

$$L^\star(f,s)=\sum_{n\ge1}\frac{\mu(n)f(n)}{n^s}=\prod_p\left(1-\frac{f(p)}{p^s}\right).$$

We can write this $L^\star$ as

$$L^\star(f,s)=\frac{1}{L(f,s)}\prod_p\left(1+\sum_{h\ge2}\frac{f(p^h)-f(p^{h-1})f(p)}{p^{hs}}\right)\ . \tag{20}$$

Denoting $f(p^h)-f(p^{h-1})f(p)$ by $f_2(p^h)$, we get that

$$L^\star(f,s)=\frac{1}{L(f,s)}\prod_p\left(1+\sum_{h\ge2}\frac{f_2(p^h)}{p^{hs}}\right):=\frac{1}{L(f,s)}\sum_{n\ge1}\frac{g(n)}{n^s}\ . \tag{21}$$

Multiplying both sides of (17) with $L^\star(f,s)$ gives the following decomposition for $\mu(\ell)f(\ell)$:

$$\mu(\ell)f(\ell)=-(\nu_r*\mu f)(\ell)+(h_r*g)(\ell)\ . \tag{22}$$

Now we consider the sum

$$S(\alpha)\ =\ \sum_{\ell\sim X}\Lambda_f(\ell)e\left(\ell\alpha\right)\ , \tag{23}$$

where $\ell\sim X$ means $X<\ell\le 2X$. Using (19) we get

$$\begin{aligned}S\left(\alpha\right)\ &=\ -\sum_{\ell\sim X}(\Lambda_f*\nu_r)(\ell)e\left(\ell\alpha\right)+\sum_{\ell\sim X}(f\log*h_r)(\ell)e\left(\ell\alpha\right)\\ &=\sum_{mn\sim X}f(n)\log(n)h_r(m)e\left(mn\alpha\right)-\sum_{\substack{mn\sim X\\ m\le M_0}}\Lambda_f(m)\nu_r(n)e\left(mn\alpha\right)\end{aligned}$$

$$- \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right)$$

for some positive real number M_0 which will be chosen later.

Let

$$L_r^{(1)}(\alpha) \ = \sum_{mn\sim X} f(n)\log(n)h_r(m)e\left(mn\alpha\right) \tag{24}$$

$$L_r^{(2)}(\alpha) \ = \sum_{\substack{mn\sim X \\ m\le M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \tag{25}$$

and call these sums as first linear sum and second linear sum respectively. With these notations we have

$$S\left(\alpha\right) \ = \ L_r^{(1)}(\alpha) - L_r^{(2)}(\alpha) - \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \ . \tag{26}$$

We call the last sum the bilinear sum. Let R be a sufficiently large real number which we will choose later. Multiplying both sides of (26) by $\frac{\mu^2(r)}{\varphi(r)}$ and summing over $1 \le r \le \frac{R}{q}$ with $(r,q)=1$ gives us

(27)

$$G_q\left(\frac{R}{q}\right) S\left(\alpha\right) = \sum_{\substack{r\le \frac{R}{q} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)}(L_r^{(1)}(\alpha) - L_r^{(2)}(\alpha)) - \sum_{\substack{r\le \frac{R}{q} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \ .$$

4. Estimating the first linear sum $L_r^{(1)}(\alpha)$

We have

$$\begin{aligned} L_r^{(1)}(\alpha) &= \sum_{mn\sim X} f(n)\log(n)h_r(m)e\left(mn\alpha\right) \\ &= \sum_{m\le 2X} h_r(m) \sum_{n\sim \frac{X}{m}} f(n)\log(n)e\left(mn\alpha\right) \ . \end{aligned}$$

Use (7) and Lemma 2.5 to get

$$\begin{aligned} L_r^{(1)}(\alpha) \ &\ll X^\eta \log X \sum_{m\le 2X} \frac{|h_r(m)|}{m^{\frac{1}{2}}} \\ &\ll X^\eta r^{1+\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} \log(X) \ . \end{aligned} \tag{28}$$

5. Estimating the second linear sum $L_r^{(2)}(\alpha)$

By definition

$$L_r^{(2)}(\alpha) \ = \sum_{\substack{m\le M_0 \\ mn\sim X}} \Lambda_f(m) \left(\sum_{d|n} \Lambda_d^{(k^2)} \right) f(n)c_r(n)e\left(mn\alpha\right) \ .$$

Since $\Lambda_d^{(k^2)} = 0$ if $d > z^{1+k^2\theta}$, the equation above can be written as

$$L_r^{(2)}(\alpha) \; = \; \sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\substack{n \sim \frac{X}{m} \\ d|n}} f(n) c_r(n) e\left(mn\alpha\right) \; .$$

By the properties of Ramanujan sum $c_r(n)$ it can be written as

$$\begin{aligned} L_r^{(2)}(\alpha) &= \sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\substack{n \sim \frac{X}{m} \\ d|n}} f(n) e\left(mn\alpha\right) \sum_{\ell | r, \ell | n} \ell \mu\left(\frac{r}{\ell}\right) \\ &= \sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell | r} \ell \mu\left(\frac{r}{\ell}\right) \sum_{\substack{n \sim \frac{X}{m} \\ d|n, \ell|n}} f(n) e(mn\alpha) \; . \end{aligned}$$

By Lemma 2.1 we get

$$\begin{aligned} L_r^{(2)}(\alpha) &= \sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell | r} \ell \mu\left(\frac{r}{\ell}\right) \sum_{n \sim \frac{X}{m[d,\ell]}} f(n[\ell,d]) e\left(mn[d,\ell]\alpha\right) \\ &= \sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell | r} \ell \mu\left(\frac{r}{\ell}\right) \\ & \qquad \sum_{n \sim \frac{X}{m[d,\ell]}} e\left(mn[d,\ell]\alpha\right) \mu([d,\ell]) \sum_{\substack{u|n \\ u|[\ell,d]^\infty}} f\left(\frac{n}{u}\right) f^{-1}([\ell,d]u) \; . \end{aligned}$$

An interchange of summation yields

$$\begin{aligned} L_r^{(2)}(\alpha) = &\sum_{m \le M_0} \Lambda_f(m) \sum_{d \le z^{1+k^2\theta}} \Lambda_d^{(k^2)} \sum_{\ell | r} \ell \mu\left(\frac{r}{\ell}\right) \\ & \qquad \mu([\ell,d]) \sum_{u|[\ell,d]^\infty} f^{-1}([\ell,d]u) \sum_{n \sim \frac{X}{mu[d,\ell]}} f(n) e\left(mnu[d,\ell]\alpha\right) \; . \end{aligned}$$

By our assumption we have

$$\begin{aligned} |L_r^{(2)}(\alpha)| &\ll X^\eta \sum_{m \le M_0} \frac{|\Lambda_f(m)|}{m^\eta} \sum_{d \le z^{1+k^2\theta}} |\Lambda_d^{(k^2)}| \sum_{\ell | r} \ell \sum_{u|[\ell,d]^\infty} \frac{|f^{-1}([\ell,d]u)|}{([\ell,d]u)^\eta} \\ &\ll X^\eta \sum_{m \le M_0} \frac{|\Lambda_f(m)|}{m^\eta} \sum_{d \le z^{1+k^2\theta}} \frac{|\Lambda_d^{(k^2)}|}{d^\eta} \sum_{\ell | r} \ell \sum_{u|[\ell,d]^\infty} \frac{|f^{-1}([\ell,d]u)|}{u^\eta} \; . \end{aligned}$$

We use Lemma 2.2 to estimate the last sum on the right hand side. This gives

$$|L_r^{(2)}(\alpha)| \ll X^\eta \sum_{m \le M_0} \frac{|\Lambda_f(m)|}{m^\eta} \sum_{d \le z^{1+k^2\theta}} \frac{|\Lambda_d^{(k^2)}|}{d^{\eta-\epsilon}} \sum_{\ell | r} \ell^{1+\epsilon} \; .$$

Since $\eta \geq \frac{1}{2}$, we get

$$|L_r^{(2)}(\alpha)| \ll X^\eta r^{1+2\epsilon} z^{(1+k^2\theta)\epsilon} \left(\sum_{m \leq M_0} \frac{|\Lambda_f(m)|}{m^{\frac{1}{2}}} \right) \left(\sum_{d \leq z^{1+k^2\theta}} \frac{|\Lambda_d^{(k^2)}|}{d^{\frac{1}{2}}} \right) . \tag{29}$$

We use the bound (1) to estimate the first sum in the brackets and Lemma 2.4 for the second sum to get

$$|L_r^{(2)}(\alpha)| \ll X^\eta r^{1+2\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} M_0^{\frac{1}{2}+\epsilon} . \tag{30}$$

6. Estimating the bilinear sum

Let

$$S_r(\alpha, M, N) \ = \sum_{\substack{mn \sim X \\ m \sim M, n \sim N}} \Lambda_f(m)\nu_r(n) e\left(mn\alpha\right) . \tag{31}$$

Let $\delta > 0$ be a sufficiently small real number which we will choose later. Let a be such that $(a,q) = 1$. We apply Lemma 2.6 with $\beta = \alpha - \frac{a}{q}$ and

$$\varphi_\ell \ = \sum_{\substack{mn = \ell \\ m \sim M, n \sim N}} \Lambda_f(m)\nu_r(n) e\left(\frac{mna}{q}\right)$$

to get

$$\begin{aligned} S_r(\alpha, M, N) \ &= \int_{-\Delta}^{\Delta} \sum_{\substack{m \sim M \\ n \sim N}} \frac{\Lambda_f(m)}{m^{iu}} \frac{\nu_r(n)}{n^{iu}} e\left(\frac{mna}{q}\right) \mathcal{H}(u) X^{iu} du \\ &\quad + \left(E_1(\delta, r) + E_2(\delta, r) + 2\delta E_3(r)\right) , \end{aligned} \tag{32}$$

where

$$E_1(\delta, r) = \sum_{\substack{X < mn \leq 2^\delta X \\ m \sim M, n \sim N}} |\Lambda_f(m)\nu_r(n)|, \tag{33}$$

$$E_2(\delta, r) = \sum_{\substack{\frac{2X}{2^\delta} < mn \leq 2X \\ m \sim M, n \sim N}} |\Lambda_f(m)\nu_r(n)|, \tag{34}$$

$$E_3(r) = \sum_{m \sim M, n \sim N} |\Lambda_f(m)\nu_r(n)|, \tag{35}$$

$$\Delta = 100 \frac{\delta^{-1} + (\beta X)^2}{\delta} . \tag{36}$$

We have the following lemma concerning the error term in (32).

Lemma 6.1.

$$E_1(\delta, r) + E_2(\delta, r) + 2\delta E_3(r) \ll \delta M \sum_{n \sim N} |\nu_r(n)| .$$

Proof. We have

$$E_1(\delta, r) = \sum_{n\sim N} |\nu_r(n)| \sum_{\substack{m\sim M \\ \frac{X}{n}<m\leq \frac{2^\delta X}{n}}} |\Lambda_f(m)|$$
$$\ll \sum_{n\sim N} |\nu_r(n)| \sum_{\frac{X}{n}\leq m\leq \frac{X}{n}+\frac{7\delta M}{5}} |\Lambda_f(m)| \ .$$

The second inequality follows since the interval $[\max(M, \frac{X}{n}), \min(2M, \frac{2^\delta X}{n})]$ is contained in $[\frac{X}{n}, \frac{X}{n} + \frac{7\delta M}{5}]$. The contribution from primes for the second sum of the above equation is $\ll \delta M$ since $|\Lambda_f(p)| \leq k \log p$ and by prime number theorem. It can be easily seen that the contribution from the higher prime powers is negligible. Hence we have

$$E_1(\delta, r) \ll \delta M \ .$$

Similarly we can show that

$$E_2(\delta, r) \ll \delta M \ .$$

The result follows since

$$E_3(r) = \left(\sum_{n\sim N} |\nu_r(n)|\right) \left(\sum_{m\sim M} |\Lambda_f(m)|\right) \ll M \left(\sum_{n\sim N} |\nu_r(n)|\right)$$

again by prime number theorem. □

Consider the sum

$$S^b = \sum_{\substack{r\leq \frac{R}{q} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \ .$$

By using Cauchy Schwarz inequality we get

$$S^b \leq \sqrt{G_q\left(\frac{R}{q}\right)} \left(\sum_{\substack{r\leq \frac{R}{q} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \left| \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \right|^2 \right)^{\frac{1}{2}} . \tag{37}$$

Consider the sum inside the brackets and call it S_1, i.e.

$$S_1 = \sum_{\substack{r\leq \frac{R}{q} \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \left| \sum_{\substack{mn\sim X \\ m>M_0}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right) \right|^2 .$$

We now examine the last sum and localize the variables m and n. Notice that $n > z$. So we start at $N = z$, go until $2z$, etc until $2^t z \leq 2X/M_0 < 2^{t+1}z$, i.e. $0 \leq t \leq \log(2X/(M_0 z))/\log 2$. Concerning M, we have $N < n \leq N' \leq 2N$, and thus $\frac{1}{2}(X/N) \leq X/n < m \leq 2X/N$. So for each N, we have two values of M, namely $M_1 = \frac{1}{2}(X/N)$ and $M_2 = X/N$.

After localizing the variables and applying Cauchy-Schwarz, we reach

$$S_1 \ll \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} \left| \sum_{\substack{mn\sim X\\ m\sim M, n\sim N}} \Lambda_f(m)\nu_r(n)e\left(mn\alpha\right)\right|^2 .$$

Using Equation (32) for the sum inside the modulus we get

$$S_1 \ll \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} \left| \int_{-\Delta}^{\Delta} \sum_{\substack{m\sim M\\ n\sim N}} \frac{\Lambda_f(m)}{m^{iu}} \frac{\nu_r(n)}{n^{iu}} e\left(\frac{mna}{q}\right) \mathcal{H}(u) X^{iu} du\right|^2$$

$$+ \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} (E_1(\delta,r)+E_2(\delta,r)+2\delta E_3(r))^2 .$$

By Lemma 6.1 the second term is

$$\ll \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} \left(\delta M \sum_{n\sim N} |\nu_r(n)|\right)^2 .$$

Hence we have

(38)

$$S_1 \ll \log\frac{2X}{M_0 z} \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} \left| \int_{-\Delta}^{\Delta} \sum_{b(q)} \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \sum_{n\sim N} \frac{\nu_r(n)}{n^{iu}} e\left(\frac{abn}{q}\right) \mathcal{H}(u) X^{iu} du\right|^2$$

$$+ \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{M,N} \left(\delta M \sum_{n\sim N} |\nu_r(n)|\right)^2 .$$

We consider the integral $I(\frac{a}{q}, M, N)$ in the above equation and after an application of Cauchy -Schwarz inequality, we obtain

$$I\Big(\frac{a}{q}, M, N\Big) \ll \int_{-\Delta}^{\Delta} \left(\sum_{b(q)} \left| \sum_{\substack{m\equiv b(q)\\ m\sim M}} \frac{\Lambda_f(m)}{m^{iu}} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{b(q)} \left| \sum_{n\sim N} \frac{\nu_r(n)}{n^{iu}} e\left(\frac{bn}{q}\right)\right|^2 \right)^{\frac{1}{2}} |\mathcal{H}(u)| du .$$

Since the sum in the first bracket inside the integral is $\ll \frac{M^2}{\varphi(q)}$ by Lemma 2.9, we get

$$I\Big(\frac{a}{q}, M, N\Big) \ll \left(\frac{M^2}{\varphi(q)}\right)^{\frac{1}{2}} \int_{-\Delta}^{\Delta} \left(\sum_{b(q)} \left| \sum_{n\sim N} \frac{\nu_r(n)}{n^{iu}} e\left(\frac{bn}{q}\right)\right|^2 \right)^{\frac{1}{2}} |\mathcal{H}(u)| du .$$

We use the Cauchy-Schwarz inequality again to get

$$I\Big(\frac{a}{q},M,N\Big) \ll \left(\frac{M^2}{\varphi(q)}\right)^{\frac12}\Big(\int_{-\Delta}^{\Delta}|\mathcal{H}(u)|^2du\Big)^{\frac12}\left(\int_{-\Delta}^{\Delta}\sum_{b(q)}\left|\sum_{n\sim N}\frac{\nu_r(n)}{n^{iu}}e\left(\frac{bn}{q}\right)\right|^2du\right)^{\frac12}.$$

Using the fact that $\int_{\mathbb{R}}|\mathcal{H}(u)|^2=\int_{\mathbb{R}}|\widehat{\mathcal{H}}(u)|^2=(2-2\delta)\frac{(\log 2)^2}{(4\pi)^2}$ leads to

$$I\Big(\frac{a}{q},M,N\Big) \ll \left(\frac{M^2}{\varphi(q)}\right)^{\frac12}\left(\int_{-\Delta}^{\Delta}\sum_{b(q)}\left|\sum_{n\sim N}\frac{\nu_r(n)}{n^{iu}}e\left(\frac{bn}{q}\right)\right|^2du\right)^{\frac12}.$$

Put this estimate in (38) to get

$$S_1 \ll \log\left(\frac{2X}{M_0z}\right)\sum_{\substack{r\le\frac{R}{q}\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{M,N}\frac{M^2}{\varphi(q)}\int_{-\Delta}^{\Delta}\sum_{b(q)}\left|\sum_{n\sim N}\frac{\nu_r(n)}{n^{iu}}e\left(\frac{bn}{q}\right)\right|^2du$$
$$+\log\left(\frac{2X}{M_0z}\right)\sum_{\substack{r\le\frac{R}{q}\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{M,N}\left(\delta M\sum_{n\sim N}|\nu_r(n)|\right)^2.$$

An application of large sieve inequality (Lemma 2.7) gives

(39)
$$S_1\ll\log\left(\frac{2X}{M_0z}\right)\sum_{M,N}\frac{M^2}{\varphi(q)}\left(\sum_{n\sim N}f^2(n)\Big(\sum_{d|n}\Lambda_d^{(k^2)}\Big)^2(n+R^2\frac{4\pi\Delta}{\log 2})\right)$$
$$+\log\left(\frac{2X}{M_0z}\right)\sum_{\substack{r\le\frac{R}{q}\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{M,N}\left(\delta M\sum_{n\sim N}|\nu_r(n)|\right)^2.$$

We assume that $R^2\delta^{-2}\ll z$ and $|\alpha-\frac{a}{q}|\ll\frac{(zR^{-2}\delta)^{\frac12}}{X}$ so that $R^2\Delta\ll z$. Hence

$$S_1\ll\log\left(\frac{2X}{M_0z}\right)\sum_{M,N}\frac{M^2N^2}{\varphi(q)}\left(\sum_{n\sim N}\frac{f^2(n)}{n}\Big(\sum_{d|n}\Lambda_d^{(k^2)}\Big)^2\right)$$
$$+\log\left(\frac{2X}{M_0z}\right)\sum_{\substack{r\le\frac{R}{q}\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{M,N}\left(\delta^2M^2N\sum_{n\sim N}|\nu_r(n)|^2\right).$$

Let $c>0$ be a small real number. Since $MN\ll X$ we get

$$S_1\ll\frac{X^2}{\varphi(q)}\log\left(\frac{2X}{M_0z}\right)\left(\frac{2X}{z}\right)^{\frac{c}{\log z}}\left(\sum_{n\le\frac{2X}{z}}\frac{f^2(n)}{n^{1+\frac{c}{\log z}}}\Big(\sum_{d|n}\Lambda_d^{(k^2)}\Big)^2\right)$$

$$+\delta^2 X^2 \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n\le \frac{2X}{z}} \frac{|\nu_r(n)|^2}{n}\,.$$

Since $|f^2(n)| \le \tau_k^2(n)$ and $\tau_k^2(n) \le \tau_{k^2}(n)$, Lemma 2.3 gives us

$$S_1 \ll \frac{X^2}{\varphi(q)} \log\left(\frac{2X}{M_0 z}\right)\left(\frac{2X}{z}\right)^{\frac{c}{\log z}} + \delta^2 X^2 \log\left(\frac{2X}{M_0 z}\right) \sum_{\substack{r\le \frac{R}{q}\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n\le \frac{2X}{z}} \frac{|\nu_r(n)|^2}{n}\,.$$

By Lemma 2.8 we have

$$S_1 \ll \frac{X^2}{\varphi(q)} \log\left(\frac{2X}{M_0 z}\right)\left(\frac{2X}{z}\right)^{\frac{c}{\log z}} + \delta^2 X^2 \log\left(\frac{2X}{M_0 z}\right)\left(\frac{2X}{z}\right)^{\frac{c}{\log z}}\left(\frac{R}{q}\right)^2. \tag{40}$$

On keeping this bound in (37), we obtain

$$S^b \le \sqrt{G_q\left(\frac{R}{q}\right)\log\left(\frac{2X}{M_0 z}\right)\left(\frac{X^2}{\varphi(q)}\left(\frac{2X}{z}\right)^{\frac{c}{\log z}} + \delta^2 X^2\left(\frac{2X}{z}\right)^{\frac{c}{\log z}}\left(\frac{R}{q}\right)^2\right)^{\frac{1}{2}}}\,. \tag{41}$$

Putting (28), (30), (41) and (27) together gives (with $\theta = 1/k^2$)

$$\begin{aligned} G_q\left(\frac{R}{q}\right) S(\alpha) \ll\ & X^{\eta} z^{(1+1)(\frac{1}{2}+\epsilon)} G_q\left(\frac{R}{q}\right)\left(\frac{R}{q}\right)^{1+2\epsilon}(\log X + M_0^{\frac{1}{2}+\epsilon}) \\ & + \frac{X}{\sqrt{\varphi(q)}}\sqrt{G_q\left(\frac{R}{q}\right)\log\left(\frac{2X}{M_0 z}\right)\left(\frac{2X}{z}\right)^{\frac{c}{\log z}}}\left(1+\delta^2\varphi(q)\left(\frac{R}{q}\right)^2\right)^{\frac{1}{2}}. \end{aligned}$$

We choose $\delta^{-1} = R = z^{\frac{1}{4}}$ to ensure that $R^2\delta^{-2} \ll z$ and $M_0 = z = X^{\frac{2}{7}(1-\eta)}$ to get

$$S(\alpha) \ll X^{\eta+\frac{1-\eta}{2}(1+\epsilon(2+\frac{1}{2}+1))}\frac{1}{q} + \frac{X}{\sqrt{\varphi(q)}}\sqrt{\frac{\log X}{G_q\left(\frac{R}{q}\right)}}\,.$$

For sufficiently small positive ϵ, the power of X in the first sum will be smaller than 1 and for the second sum we use (10) and (12) (with $q \le R^{14/15} = X^{(1-\eta)/15}$) together to get

$$S(\alpha) \ll \frac{\sqrt{q}X}{\varphi(q)}\,.$$

This completes the proof of Theorem 1.4.

7. Proof of Theorem 1.5

Since the proof of Theorem 1.5 is similar to that of Theorem 1.4, we only give a sketch of the proof.

By (22), we have the following decomposition:

$$\sum_{\ell\sim X} \mu(\ell) f(\ell)(\ell) e(\ell\alpha) \ =\ L_r^{(1)}(\alpha) - L_r^{(2)}(\alpha) - \sum_{\substack{mn\sim X\\ m>M_0}} \mu(m) f(m)\nu_r(n) e(mn\alpha) \tag{42}$$

where

$$L_r^{(1)}(\alpha) = \sum_{\ell \sim X} (h_r * g)(\ell) e(\ell\alpha),$$

$$L_r^{(2)}(\alpha) = \sum_{\substack{mn \sim X \\ m \leq M_0}} \mu(m) f(m) \nu_r(n) e(mn\alpha).$$

We use $|g(n)| \ll \log^2 n$, g is supported only on square-full integers and the number of such integers upto X is $\ll X^{1/2}$ while estimating the sum $L_r^{(1)}$. We can get

$$L_r^{(1)}(\alpha) \ll X^{\frac{1}{2}} r^{1+\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} \log^2 X \ .$$

The sum $L_r^{(2)}(\alpha)$ can be estimated exactly as in Section 5, where one has to use the bound

$$\sum_{m \leq M_0} \frac{|f(m)|}{m^{\frac{1}{2}}} \ll M_0^{\frac{1}{2}+\epsilon}$$

which follows from the bound $|f(m)| \ll m^\epsilon$.

$$|L_r^{(2)}(\alpha)| \ll X^\eta r^{1+2\epsilon} z^{(1+k^2\theta)(\frac{1}{2}+\epsilon)} M_0^{\frac{1}{2}+\epsilon} \ . \tag{43}$$

We can estimate the third sum on the right hand side as in Section 6 with some modifications which we will write down here. The corresponding bound for the bound in Lemma 6.1 will be

$$E_1(\delta, r) + E_2(\delta, r) + 2\delta E_3(r) \ll \sqrt{W_f(X)} \delta M \sum_{n \sim N} |\nu_r(n)| \ ,$$

when using (5) . Instead of Lemma 2.9 one has to use

$$\sum_{b (\mathrm{mod}) q} \left| \sum_{\substack{m \equiv b(q) \\ m \sim M}} \frac{f(m)}{m^{iu}} \right|^2 \ll \frac{W_f(X) M^2}{q} \ . \tag{44}$$

With these changes and choosing parameters exactly as in Section 6, we get the mentioned result.

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